Eur. Phys. J. D **18**, 371–377 (2002)
DOI: 10.1140/epjd/e20020043

THE EUROPEAN PHYSICAL JOURNAL D

EDP Sciences

Series solution to the laser-ion interaction in a Raman-type configuration

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Received 12 March 2001 and Received in final form 16 October 2001

Abstract. The Raman interaction of a ultracold ion trapped with two travelling wave lasers is studied analytically with series solutions, in the absence of the rotating wave approximation (RWA) and without restrictions of both the Lamb-Dicke limit and the weak excitation regime. The comparison is made between our solutions and those obtained under the RWA in order to demonstrate the validity region of the RWA. As a practical example, the preparation of Schrödinger-cat states is proposed beyond the weak excitation regime, using our calculations.

PACS. 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements – 32.60.+i Zeeman and Stark effects

1 Introduction

The preparation of ultracold ions as well as the production of nonclassical motional states for these ions plays a central role in ion trap experiments [1–3]. As the trapped ion system is in near perfect isolation, immune from the violation of the environment, and at the same time the electric-dipole forbidden transition is usually introduced to avoide the dissipation of the ion, decoherence in the preparation and preservation of nonclassical states can be effectively suppressed. Consequently, the ion trap system is also considered to be a promising tool for quantum computing [4–6].

The various theoretical schemes for generating the nonclassical states of motion of the trapped ions and achieving quantum computing with trapped ions are based on the Jaynes-Cummings (JC) model [7], in which the ion is assumed to behave two levels, the trap's potential follows that of a quantized harmonic oscillator, and the radiating lasers are considered to be classical forms of standing or travelling waves. The general consideration is taken for the case of Lamb-Dicke limit (LDL) under the weak excitation regime (WER), which corresponds to the actual case in present ion trap experiments [3]. The LDL means that the ions move within the region of the space much smaller than the effective wavelength of the laser, and the WER means that the laser-ion interaction is much smaller than the trap frequency. For this case, some techniques developed in the framework of cavity QED based on the JC model can be immediately transcribed to the ion trap system by taking advantage of the analogy between the cavity QED and ion trap problem. In the case of motion of ions exceeding the LDL, we can use the technique proposed in reference [8]. In a word, under the rotating-wave approximation (RWA), the problem of trapped ions interacting with lasers is solvable. However, the strength of the coupling between the ions and lasers can be conveniently adjusted simply by changing the intensity of lasers. If the coupling strength is much larger than the trap frequency, called the strong excitation regime (SER), then the RWA is no longer applicable since the rapidly oscillating (*i.e.*, counter-rotating) terms also make a significant contribution to the interaction, which is very different from the case in Cavity QED [9]. It has been reported that the SER is useful for preparing nonclassical motional states of a trapped ion and implementing quantum computing with trapped ions [6,10–12]. For example in reference [10], Schrödinger cat states could be prepared readily in the SER. Based on that work, a proposal was made for motional state engineering and endoscopy in the SER [11]. As the quantum state is sensitive to decoherence, the manipulation of trapped ions in the SER is of experimental importance because of the great reduction of the operation time. For the same reason, a scheme for hot-ion quantum computing also requires the condition of the SER [6]. In contrast to the theoretical studies, the ion-laser interaction in the SER has not yet been reported experimentally since the high-laser intensity produces increased off-resonant

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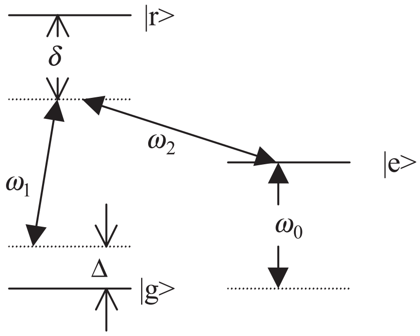

Fig. 1. Level scheme of the internal structure of the trapped ultracold ion, where $|g\rangle \leftrightarrow |e\rangle$ is dipole forbidden.

excitation of carrier transitions which limits the final ground state occupation achieved by the cooling process [13]. The strong laser also modifies the original electronic levels of the ions [14], making the problem somewhat complicated. In a numerical study of the strong Raman sideband excitation driving the vibrational ground state of the trapped ion [15], it is shown that the non-RWA interaction terms produce some limitations for two strong laser beams detuned by the vibrational frequency to be used as Raman motional displacement beams, whereas the interaction is useful for the preparation of the Schrödinger cat state.

Generally speaking, as the RWA is invalid, the ion trap problem in the SER can not be solved analytically with the JC model. So some approximations have to be introduced as in references [10,11] and the studies in references [12,15] with resort to numerical calculations. In fact, as shown in reference [16], even if the ions are governed by the WER, in which the RWA is valid, the deletion of the RWA can also present some interesting results very different from those obtained under the RWA. In that work, some particular analytical solutions have been obtained in the absence of the RWA for large detuning cases, and Schrödinger cat states could be readily prepared. However, that work is too simple. We hope to investigate the Raman configuration which has been extensively applied in actual ion trap experiments [3], instead of the simple case in reference [16]. We also hope to exclude the assumption related to both the WER and LDL. Hence, in this contribution, we attempt to undertake a more general and complete investigation of the trapped ion under the Raman process. Moreover, a comparison of our results with those under the RWA is made. In addition, we propose a scheme for preparing the Schrödinger cat state beyond the WER.

2 Model and solution

We study a single ultracold ion radiated by lasers in the Raman-Λ-type configuration, as shown in Figure 1. The electronic structure shown is employed with two lower levels $|e\rangle$ and $|g\rangle$ coupled to a common upper state $|r\rangle$, and the two lasers with frequencies ω_1 and ω_2 respectively are assumed to propagate along opposite directions. For a sufficiently large detuning to the level $|r\rangle$, $|r\rangle$ may be adiabatically eliminated. Then we have an effective two-level system, in which the lasers drive the electric-dipole forbidden transition $|g\rangle \leftrightarrow |e\rangle$. The dimensionless Hamiltonian of such a system in the frame rotating with the effective laser frequency ω_l $(= \omega_1 - \omega_2)$ can be written as [10]

$$H = \frac{\Delta}{2}\sigma_z + a^+a + \frac{\Omega}{2}\left(\sigma_+ \mathrm{e}^{\mathrm{i}\eta\hat{x}} + \sigma_- \mathrm{e}^{-\mathrm{i}\eta\hat{x}}\right) \tag{1}$$

where the detuning $\Delta = (\omega_0 - \omega_l)/\nu$ with ω_0 being the optical resonance frequency, *i.e.*, the transition frequency of two levels of the ion, and ν the frequency of the trap. Ω is the dimensionless Rabi frequency and η the effective Lamb-Dicke parameter given by $\eta = \eta_1 + \eta_2$ with subscripts denoting the counterpropagating laser fields. σ_i $(i = \pm, z)$ are Pauli operators, $\hat{x} = a^+ + a$ is the dimensionless position operator of the ion with a^+ and a being operators of creation and annihilation of the phonon field, respectively. The notations "+" and "−" in front of $\mathrm{i}\eta\hat{x}$ indicate the absorption of a photon from one beam followed by emission into the other beam and *vice versa*, respectively. ν is generally supposed to be much greater than the atomic decay rate, called the strong confinement limit, neglecting the effect of the atomic decay. In general, one may expand $\mathrm{e}^{\pm\mathrm{i}\eta\hat{x}}$ in equation (1) to the first-order terms of $\eta\hat{x}$ and neglect other higher-order terms by supposing the ion is governed by the WER ($\Omega \ll 1$) and within the LDL ($\eta \ll 1$). However, as we hope to treat the problem more exactly and generally, we first perform following unitary transformations on equation (1), that is

$$\begin{aligned} H^I = V^+U^+HUV = \frac{\Omega}{2}\sigma_z + a^+a \\ + g\left(a^+ + a\right)(\sigma_+ + \sigma_-) + \epsilon(\sigma_+ + \sigma_-) + g^2 \end{aligned} \tag{2}$$

where

$$U = \begin{pmatrix} D(\beta) & -D(\beta) \\ D^+(\beta) & D^+(\beta) \end{pmatrix}$$

and $V = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi a^+ a/2}$ with $D(\beta) = \mathrm{e}^{\mathrm{i}\eta(a^+ + a)/2}$, $g = \eta/2$ and $\epsilon = -\Delta/2$. Actually, the unitary transformation U made above is identical to that made in [17]. But to coincide with the standard form of the non-RWA model, V is performed. Comparing with the standard non-RWA JC model [9], there exists an additional driven term and an additional constant term in equation (2), and the optical resonance frequency is replaced by the Rabi frequency. In what follows, we will adopt the coherent state representation [18], by which the non-RWA JC models have been analytically treated [16,19]. Thus equation (2) is rewritten as

$$\begin{aligned} H = \frac{\Omega}{2}\sigma_z + \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha} + g\left(\alpha + \frac{\mathrm{d}}{\mathrm{d}\alpha}\right)(\sigma_+ + \sigma_-) \\ + \epsilon(\sigma_+ + \sigma_-) + g^2 \end{aligned} \tag{3}$$

where α is a complex number, and

$$\int \frac{\mathrm{d}\alpha\mathrm{d}\alpha^*}{2\pi\mathrm{i}} \exp\left(-|\alpha|^2\right) |\alpha^*\rangle\langle\alpha^*| = 1$$

$$\frac{2g^2-\left(1-2g^2\pm 2\epsilon\right)\left(2+\Omega/2-2g^2\pm 2\epsilon\right)+\Omega+\Omega^2/4}{2g^2-\left(1-2g^2\pm 2\epsilon\right)\left(2-\Omega/2-2g^2\pm 2\epsilon\right)-\Omega+\Omega^2/4}= \frac{4g^2\left(2+3\Omega/4-2g^2\pm 2\epsilon\right)-\Omega\left(g^2\mp\epsilon\right)\left(3\pm 2\epsilon+\Omega-2g^2\right)}{4g^2\left(2-3\Omega/4-2g^2\pm 2\epsilon\right)+\Omega\left[\left(1+\Omega/2\right)\left(2-\Omega/2\right)+2\left(g^2\mp\epsilon\right)^2-3\left(g^2\mp\epsilon\right)\right]}\cdot \tag{7}$$

with the relation between the coherent state and Fock state

$$|n\rangle=\int\frac{\mathrm{d}\alpha\mathrm{d}\alpha^*}{2\pi\mathrm{i}}\exp\left(-|\alpha|^2\right)|\alpha^*\rangle\frac{1}{\sqrt{n!}}\alpha^n. \tag{4}$$

Using the same idea as in references [16,19], we assume the eigenfunction of the Schrödinger equation of equation (3) takes the series form

$$\Psi(\alpha)=\begin{pmatrix}\Psi_1(\alpha)\\ \Psi_2(\alpha)\end{pmatrix}=\begin{pmatrix}\exp(-z\alpha)\sum_{n=0}^{\infty}b_n\alpha^n\\ \exp(-z\alpha)\sum_{n=0}^{\infty}c_n\alpha^n\end{pmatrix}$$

where z, b_n and c_n are constants determined later. Taking $\Psi(\alpha)$ into the Schrödinger equation of equation (3) will yield two recurrence relations

$$b_{n+1}=\frac{1}{g(n+1)}\left[\left(E+\frac{\Omega}{2}-n-g^2\right)c_n + (gz-\epsilon)b_n-gb_{n-1}+zc_{n-1}\right], \tag{5}$$

$$c_{n+1}=\frac{1}{g(n+1)}\left[\left(E-\frac{\Omega}{2}-n-g^2\right)b_n + (gz-\epsilon)c_n-gc_{n-1}+zb_{n-1}\right] \tag{6}$$

where E is the trial solution of the eigenenergy of equation (3). As the solution procedure is similar to that found in reference [16], in what follows, we just list the results:

1. $b_n=c_n=0$ for $n\geq 2$, we have two solutions. One is $E_1=1+\epsilon$ with E_1 the eigenenergy of the system in this case, corresponding to $z=g$ and a restricted relation of $\Omega=2\sqrt{1+2\epsilon-4g^2}$. The other is $E_1'=1-\epsilon$, with $z=-g$ and a restricted relation of $\Omega=2\sqrt{1-2\epsilon-4g^2}$. The eigenfunctions for the above two cases are respectively

$$\Psi(\alpha)=\mathrm{e}^{-g\alpha}\begin{pmatrix}b_0+b_1\alpha\\ c_0+b_1\alpha\end{pmatrix}$$

and

$$\Psi'(\alpha)=\mathrm{e}^{g\alpha}\begin{pmatrix}b_0'+b_1'\alpha\\ c_0'-b_1'\alpha\end{pmatrix},$$

where

$$c_0=\frac{1-\Omega/2+2\epsilon-2g^2}{1+\Omega/2+2\epsilon-2g^2}b_0,$$

$$b_1=\frac{1}{g}\left[\left(1+\frac{\Omega}{2}-g^2+\epsilon\right)c_0+\left(g^2-\epsilon\right)b_0\right],$$

$$c_0'=-\frac{1-\Omega/2-2\epsilon-2g^2}{1+\Omega/2-2\epsilon-2g^2}b_0',$$

and

$$b_1'=\frac{1}{g}\left[\left(1-\epsilon-g^2+\frac{\Omega}{2}\right)c_0'-\left(g^2+\epsilon\right)b_0'\right]\cdot$$

b_0 and b_0' can be set to be 1 for normalisation;

2. $b_n=c_n=0$ for $n\geq 3$, the two solutions are $E_2=2\pm\epsilon$ corresponding to $z=\pm g$ and two independent restricted conditions, which can be solve from the following equation,

see equation (7) above.

The eigenfunctions at this moment in the unit of $b_0=b_0'=1$ are, respectively,

$$\Psi(\alpha)=\mathrm{e}^{-g\alpha}\begin{pmatrix}1+b_1\alpha+b_2\alpha^2\\ c_0+c_1\alpha+b_2\alpha^2\end{pmatrix}$$

and

$$\Psi'(\alpha)=\mathrm{e}^{g\alpha}\begin{pmatrix}1+b_1'\alpha+b_2'\alpha^2\\ c_0'+c_1'\alpha-b_2'\alpha^2\end{pmatrix}$$

where

$$c_0=\frac{\left(1-2g^2+2\epsilon\right)\left(2-\Omega/2-2g^2+2\epsilon\right)+\Omega-\Omega^2/4-2g^2}{\left(1-2g^2+2\epsilon\right)\left(2+\Omega/2-2g^2+2\epsilon\right)-\Omega-\Omega^2/4-2g^2},$$

$$c_1=\frac{1}{g}\left[2+\epsilon-g^2-\frac{\Omega}{2}+\left(g^2-\epsilon\right)c_0\right],$$

$$b_1=\frac{1}{g}\left[\left(2+\epsilon-g^2+\frac{\Omega}{2}\right)c_0+g^2-\epsilon\right],$$

$$b_2=\frac{1}{2g^2}\left\{\left(g^2-\epsilon\right)\left(3-\Omega+2\epsilon-2g^2\right)+g^2 +\left[\left(1+\epsilon-g^2\right)^2-1-\frac{\Omega^2}{4}-\frac{\Omega}{2}-g^2-\left(g^2-\epsilon\right)^2\right]c_0\right\},$$

$$c_0'=\frac{2g^2-\left(1-2g^2-2\epsilon\right)\left(2-\Omega/2-2g^2-2\epsilon\right)-\Omega+\Omega^2/4}{\left(1-2g^2-2\epsilon\right)\left(2+\Omega/2-2g^2-2\epsilon\right)-\Omega-\Omega^2/4-2g^2},$$

$$c_1'=\frac{1}{g}\left[2-\epsilon-g^2-\frac{\Omega}{2}-\left(g^2+\epsilon\right)c_0'\right],$$

$$b_1'=\frac{1}{g}\left[\left(2-\epsilon-g^2+\frac{\Omega}{2}\right)c_0'-g^2+\epsilon\right],$$

and

$$b_2'=\frac{1}{2g^2}\left\{\left(g^2+\epsilon\right)\left(3-\Omega-2\epsilon-2g^2\right)+g^2 +\left[\left(1-\epsilon-g^2\right)^2-1-\frac{\Omega^2}{4}-\frac{\Omega}{2}-g^2-\left(g^2+\epsilon\right)^2\right]c_0'\right\};$$

3. when the above technique is extended to the case of $b_n = c_n = 0$ for $n \geq N+1$, the series solution becomes very complicated. However the eigenenergy of the system is still of the simple form, that is, $E_N = N \pm \epsilon$, corresponding to $z = \pm g$ and two independent sets of much complicated expressions of restricted conditions which we have to omit here.

3 Discussion

As limitations for both the Rabi frequency and Lamb-Dicke parameter as well as the RWA are excluded in the present treatment, the results we obtained above are more exact and general than those depending on the RWA or the approximate expansion of small values of η and Ω. However, as the problem involving the counter-rotating terms is non-integrable [16,18,19], the present solutions obtained by the termination of the series of the trial solution $\Psi(\alpha)$ are not general and are only valid under some restricted conditions related to some parameters of the system. Nevertheless, with the present solutions, we can investigate certain cases over a wide range of parameters, particularly beyond the WER and LDL. It is also of interest to make a comparison of our results with those obtained under the RWA. Let us first consider a special case with both $\Omega \sim \epsilon \geq 1$ and $\eta \to 0$. In such a case, equation (3) is reduced to

$$H^{\mathrm{s}} = \frac{\Omega}{2}\sigma_z + \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha} + \epsilon(\sigma_+ + \sigma_-).$$

Repeating the above procedure of the series solution for the simplest case, *i.e.*, $b_n = c_n = 0$ for $n \geq 2$, we can find $z = 0$ and

$$\Psi^{\mathrm{s}} = \begin{pmatrix} b_0 + b_1\alpha \\ c_0 + c_1\alpha \end{pmatrix}$$

with

$$c_0 = \frac{E^{\mathrm{s}} - \Omega/2 - 1}{\epsilon} b_0,$$

$$c_1 = \frac{E^{\mathrm{s}} - \Omega/2 - 1}{\epsilon} b_1$$

and

$$E^{\mathrm{s}} = 1 \pm \sqrt{\epsilon^2 + \frac{\Omega^2}{4}}\cdot$$

We can easily find that the eigenenergy E^{s} is in good agreement with the solution for a Fock state representation for $n = 1$. However, to make a comparison for eigenfunctions with the former solutions, we have to return to the original representation before equation (2). The eigenfunction in the original representation is

$$\Psi^{\mathrm{os}} = UV\Psi^{\mathrm{s}} \approx \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\Psi^{\mathrm{s}}(\alpha)\mathrm{e}^{-\mathrm{i}\pi a^+ a/2}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} (b_0 - c_0)|0\rangle - \mathrm{i}(b_1 + c_1)|1\rangle \\ (b_0 + c_0)|0\rangle - \mathrm{i}(b_1 + c_1)|1\rangle \end{pmatrix} \quad (8)$$

where we use the fact [16,18] that $\mathrm{e}^{g\alpha}\alpha^n$ in a coherent state representation corresponds to the displaced Fock states $|n, g\rangle$ as given by equation (4) and the Baker-Campbell-Hausdorff formula. Therefore, in the case of $\Omega \ll 1$ but $\epsilon \gg 1$, we have from equation (8)

$$\Psi^{\mathrm{os}} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which means no transitions existing in the large detuning case. If we assume $\epsilon \to 0$ and Ω is an arbitrary positive real number, we have

$$\Psi^{\mathrm{os}} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which corresponds to the carrier excitation. Both solutions above are also in good agreement with the solutions in a Fock state representation [12]. However, from equation (8) we can know that, unlike the former solutions for the WER case, the probability amplitudes of up and down states of the ion are different and complicated in the case of the SER with large detunings. This is a new result which has not been obtained before by general approaches made in a Fock state or dressed state representation.

For a more general comparison, we should first present the solutions under the RWA. As referred to in Section 1, equation (1) can be treated as a nonlinear JC model with the approach proposed in reference [8] under the RWA. However, to make the comparison clearer and simpler, we start our RWA treatment from equation (2). By performing a unitary transformation $\exp[-\mathrm{i}(\frac{\Omega}{2}\sigma_z + \frac{a^+a}{2^M})t]$ on equation (2) with $M = 1, 2, 3, ...$, we can obtain

$$H_M = \left(1 - 2^{-M}\right)a^+a + g\left(a^+\sigma_- + a\sigma_+\right) + g^2 \quad (9)$$

corresponding to the resonance conditions of $\Omega = 2^{-M}$ under the RWA, and the eigenenergy in a Fock state representation is of the form

$$E_M^{\pm} = \left(1 - 2^{-M}\right)\left(n + \frac{1}{2}\right) + \frac{\eta^2}{4}$$
$$\pm \frac{1}{2}\sqrt{\eta^2(n+1) + \left(1 - 2^{-M}\right)^2} \quad (10)$$

with $n = 0, 1, 2, ...$ Meanwhile, performing a unitary transformation of $\exp[-\mathrm{i}(\frac{\Omega}{2K}\sigma_z + \frac{a^+a}{2})t]$ on equation (2) with $K = 1, 2, 3, ...$, under the RWA resonance conditions of $\Omega = K$, we have

$$H_K = \frac{K-1}{2K}\Omega\sigma_z + g\left(a^+\sigma_- + a\sigma_+\right) + g^2 \quad (11)$$

whose eigenenergy in a Fock state representation is of the form

$$E_K^{\pm} = \frac{\eta^2}{4} \pm \frac{1}{2}\sqrt{\eta^2(n+1) + (K-1)^2} \quad (12)$$

with $n = 0, 1, 2, ...$

As shown in the last section, although our method can in principle present all series solutions, we merely present the specific forms of the non-RWA solution for the two

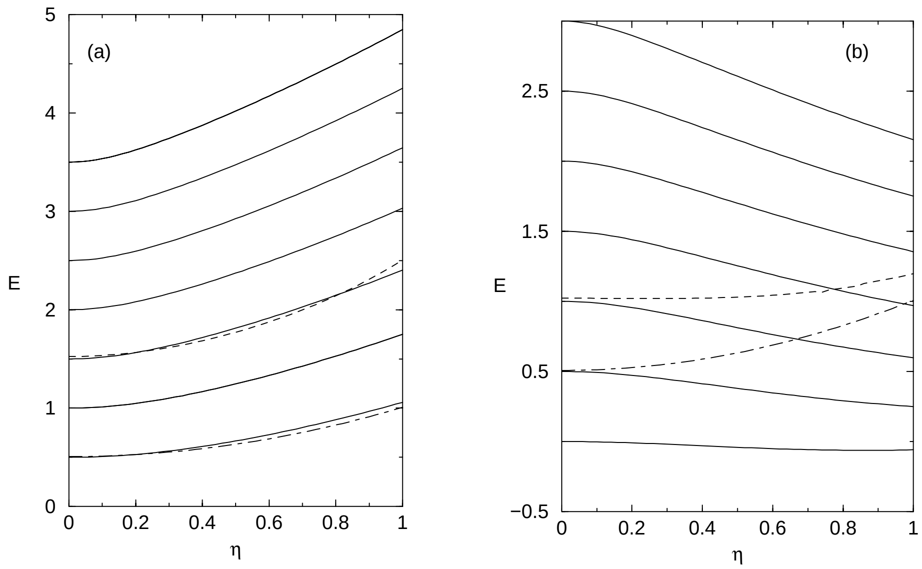

Fig. 2. Variation of E with respect to η in the case of $\Omega = 0.5$, where solid curves from the bottom to top are RWA solutions of equation (10), corresponding to $n = 0$, 1, 2, 3, 4, 5 and 6, respectively, dot-dashed curve is the solution of equation (13) and dashed curves are from equation (A.3) or equation (A.4) (because the solutions of Eqs. (A.3, A.4) are identical); (a) demonstrates E^+, and (b) is for E^-.

simplest cases. Here we will use those two cases for a comparison with equations (10, 12). For case 1, we find that the two solutions for $E = 1 \pm \epsilon$, along with two different restricted relations respectively, actually correspond to the same expression

$$E = \frac{1}{2} + \frac{1}{2}\eta^2 + \frac{1}{8}\Omega^2. \tag{13}$$

However, for case 2, the situation is somewhat complicated. The direct algebra shown in Appendix presents four solutions of E for this case, whereas the specific calculation shows that the solutions of equations (A.3, A.4) are actually identical.

In Figures 2 and 3, the solutions under the RWA are compared with those for the non-RWA treatment in the case of $\Omega = 0.5$ and 3.0. We can find that, in the case of small values of Ω and η, some of the solutions under the RWA can approach those without the RWA. It is physically reasonable because the RWA is only valid for the WER and LDL. However, although the solutions without the RWA are exact, only a few of these solutions can be obtained for a certain termination of the series solution. We can not obtain all the particular solutions of the system unless we make the termination of the series solution for almost infinite times up to the case with infinite series terms. In contrast, the RWA treatment can present general solutions, like equations (10, 12). Nevertheless, as it merely retains some resonance terms in the Hamiltonian, a specific solution under the RWA only corresponds to a specific value of Ω. Moreover, the figures tell us that, only when the value of η is not taken to be larger than 0.1, can we consider that the solutions under the RWA and without the RWA are in good agreement in the case of the WER. The RWA description is obviously inaccurate for the case beyond the LDL although Ω is less than 1.0 in this case.

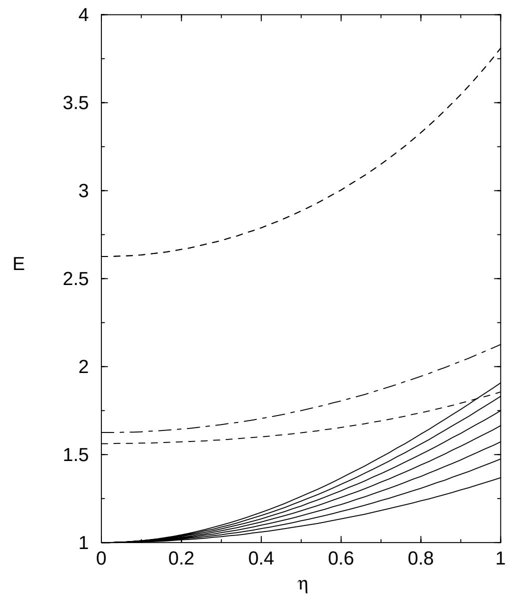

Fig. 3. Variation of E with respect to η in the case of $\Omega = 3.0$, where solid curves from the bottom to top are RWA solutions of equation (12) for the case of E^+, corresponding to $n = 0$, 1, 2, 3, 4, 5 and 6, respectively, the dot-dashed curve is the solution of equation (13) and dashed curves are from equation (A.3) for cases of $E^{\pm}$ respectively.

When the value of Ω is larger, the difference between the RWA case and non-RWA one becomes great, as shown in Figure 3. It is difficult to find any correspondence for the two cases. However, there are some crossing points between the RWA solutions and non-RWA ones. This means at certain values of η, the solutions for eigenenergies in these two cases can be the same. In physics, it can be considered that the non-RWA solutions are some isolated ones, merely corresponding to those of up internal levels of the ion. One may not expect to obtain the general eigenenergies with our non-RWA treatment. However, the demonstration of above series solutions means the existence of the finite dimensional invariant subspace of some operators. Particularly from the simple forms of the eigenenergies, one may expect that the quantum KAM theorem should be applicable to the SER problem [20].

Before ending our discussion of the usefulness of non-RWA solutions, it is also of interest to present a practical example of preparing the nonclassical motional states of the ion with our solutions. As referred to in Section 1, there is a considerable interest in investigating the SER, particularly to rapidly prepare nonclassical motional states of the ion. In fact, with the results in the present paper, one can also generate some particular nonclassical states through suitable adjustment of the Rabi frequencies and proper measurements. For example, consider the simplest case, *i.e.*, the case 1 in Section 2. For the case of the resonance ($\epsilon = 0$), we have $E_1 = E_1'$ with the same Ω and double degenerate eigenfunctions $\Psi(\alpha)$ and $\Psi'(\alpha)$. Transforming our results into the Schrödinger representation yields the states $\Phi(t) = \exp(-\mathrm{i}E_1 t + \mathrm{i}\omega_1\sigma_z t/2)UV\Psi(\alpha)$ and $\Phi'(t) = \exp(-\mathrm{i}E_1 t + \mathrm{i}\omega_1\sigma_z t/2)UV\Psi'(\alpha)$, respectively. In the case of the LDL ($g \ll 1$), the Rabi frequency Ω would be $2\sqrt{1-4g^2} \sim 2$. So we have

$$c_0 \sim \frac{g^2}{g^2-1}b_0,$$

$$b_1 \sim \frac{g}{g^2-1}b_0,$$

$$c_0' \sim \frac{g^2}{g^2-1}b_0'$$

and

$$b_1' \sim \frac{g}{g^2-1}b_0'.$$

By controlling the evolution time $t = 4\pi/\omega_1$, and measuring the excited state of the ion, we can obtain a displaced even coherent state, a kind of Schrödinger-cat states

$$\Phi^M \sim \frac{1}{\sqrt{2}}D(\beta)\left(|\mathrm{i}\frac{\eta}{2}\rangle + |-\mathrm{i}\frac{\eta}{2}\rangle\right) = \frac{1}{\sqrt{2}}\left(|\mathrm{i}\eta\rangle + |0\rangle\right) \quad (14)$$

where the relation between the coherent state and Fock state has been used. As the Rabi frequency approaches 2, such a preparation is obviously outside the WER.

4 Conclusion

As $\mathrm{e}^{g\alpha}\alpha^n$ corresponds to the displaced Fock states $|n, g\rangle$, the eigenfunctions of the system we obtained are infinite superposition of displaced Fock states. Therefore, it is understandable that we can not obtain similar analytical results by general Fock state expansion. The obvious advantage of our approach is the possibility to obtain some relations between parameters of the system corresponding to certain nonclassical states by truncating the series expansion step by step. So as long as we can reach these conditions experimentally, the different nonclassical states predicted by our theory would be obtained.

In summary, the Raman interaction between a trapped ultracold ion and two travelling wave lasers has been treated analytically in an interaction representation, without consideration of the RWA and the limitation for both the LDL and WER. Although the coherent state expansion technique was used previously for treating a similar problem, and the solutions we presented here are some isolated ones under certain conditions related to Ω, η and Δ, the present investigation is more general and exact. Hence we can study the ion trap problem in a wide range of parameters and compare our results with other approximate works or numerical solutions in this respect as we did in the present paper and for the cavity-atom problem [21]. More importantly, the Raman process has been widely used for the laser-cooling in current ion trap experiments, and the work outside the WER is attracting much interest from experimentalists. One possible way [10] to experimentally realizing the SER is that the ion is first cooled within the LDL and under the WER, and then the trap frequency is decreased by opening the trap adiabatically so that the ratio of the Rabi frequency to the trap frequency is increased to a large number. Therefore, we believe that the difficulties in realizing the SER will soon be overcome, and our work would be helpful for any possible future exploration of the quantum properties of the ion-trap system outside the WER.

The author sincerely thanks M.A. Kornberg for his carefully reading of the manuscript. The work is partly supported by National Natural Science Foundation of China.

Appendix

To obtain a more specific expression of the eigenenergy for case 2 in Section 2, we can suppose $X = \epsilon - g^2$ for the case of $z = g$ and $Y = \epsilon + g^2$ for $z = -g$. Direct algebra on equation (7) can present the following two second-order differential equations

$$AX^2 + BX + C = 0 \quad (A.1)$$

and

$$AY^2 - BY + C = 0 \quad (A.2)$$

with

$$A = 8(1-g^2),$$

$$B = \left(3+\frac{\Omega}{2}\right)(2+\Omega)\left(2-\frac{\Omega}{2}\right) - 3\left(\frac{\Omega}{2} - 2 + \frac{\Omega^2}{4} + 2g^2\right) - 28g^2 - (3+\Omega)\left(2+\frac{\Omega}{2} - \frac{\Omega^2}{4} - 2g^2\right)$$

and

$$C = -\left[\left(\frac{\Omega}{2} - 2 + \frac{\Omega^2}{4} + 2g^2\right)\left(1 + \frac{\Omega}{2}\right)\left(2 - \frac{\Omega}{2}\right)\right.$$
$$\left. +20g^2 - 6g^2\left(\frac{\Omega^2}{4} + 2g^2\right)\right].$$

So we can obtain the eigenenergies

$$E_1^{\pm} = 2 + g^2 + \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{A.3}$$

for the case of $z = g$ and

$$E_2^{\pm} = 2 + g^2 - \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{A.4}$$

for the case of $z = -g$.

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